

Statistics & Probability

Use Probability to Evaluate Outcomes of Decisions

Introduction

This learning progression is designed for a high school algebra II class (typically 10th– 11th grade). The textbook used in the class is McDougal Littell’s *Algebra 2*, 2004. The CCSS Math cluster focused on in this progression is “Use Probability to Evaluate Outcomes of Decisions”^{S-MD.B} under the statistics and probability domain.

When students learn the mathematical structure of statistics and probabilities, they begin to realize that the informal statistical inferences often made when making a decision, can be made more precise using mathematics. By using a combination of conceptual, procedural, and modeling teaching techniques, students can learn how to use raw statistical data to make decisions. It is important that students also understand that probabilities and statistics should be thought of as long-term relative frequencies, not just on a case-by-case basis. Students should use this concept to influence their decision making.

Statistics and probability can also be an influential subject in a mathematics course because it often has direct connections with real world applications. Jobs in actuarial fields, manufacturing, and production are all fairly high-profile jobs that involve complex applications of statistical data. Students may connect with this kind of mathematics more than other concepts simply because they can see a direct connection with careers and college.

Domain: Statistics & Probability

Cluster: Using Probability to Evaluate Outcomes of Decisions

Standards:

CCSS.Math.Content.HSS-MD.B.5

Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values.

CCSS.Math.Content.HSS-MD.B.5a

Find the expected payoff for a game of chance.

CCSS.Math.Content.HSS-MD.B.5b

Evaluate and compare strategies on the basis of expected values.

CCSS.Math.Content.HSS-MD.B.6

Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).

CCSS.Math.Content.HSS-MD.B.7

Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game)

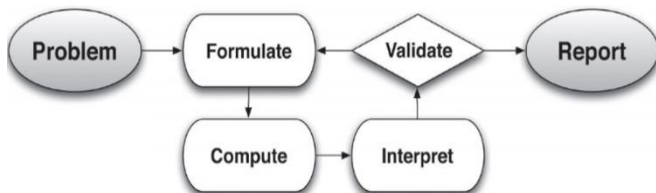
HSS-MD.B.5, HSS-MD.B.5a

People naturally want to have any kind of advantage possible when making decisions or playing a game. The first step in making decisions based on statistics and probability is to assign probabilities to payoff values and find expected values.^{S-MD.B.5} Once these values are established, then the expected payoff for a game of chance can be found^{S-MD.B.5a}.

The two standards that address the process above are interrelated and can be taught almost simultaneously. First, by using simple, contrived numerical examples, students learn to calculate the expected values of a certain outcome. An example problem, such as finding the expected winnings from a lottery ticket*. Students can use the expected value equation $E(x) = x_1p_1 + x_2p_2 + \dots + x_ip_i$ to find the probability of an expected outcome.

One way to model this kind of mathematical thinking is through classic casino games such as “craps” or “blackjack.” To conduct this activity, once the traditional rules of the games are explained, students can work in small groups or at stations. One student can take on the role of the dealer and can rotate through so every student has the chance to be the dealer and play. Additionally, students should keep track of their calculations for each hand/roll and to put the mathematics into practice. This way, as the teacher monitors the activity, s/he can ask a student to explain the probability of succeeding in their next turn and to show, on paper, how they arrived at that answer. Students must use fractions and ratios to assign expected values for an outcome.

This kind of modeling activity requires students to use the basic modeling cycle. Students can be taught the process of (1) identifying variables (the number of players, for example), (2) formulating a model by creating and selecting statistical representations that describe the relationships between variables, (3) analyzing and performing operations on the relations to draw conclusions (playing the game), (4) interpreting the results, (5) validating the conclusions by comparing them with similar situations, improving the methods if needed, and (6) reporting the results.



*Example Problem: Lottery Ticket

To win the state lottery, you must match 5 different numbers, in any order, chosen from 1 – 49, plus another number chosen from 1 – 42. The jackpot is \$45,000,000.

- How many different tickets would you need to buy to ensure that you would win?
- How many different tickets would you have to buy to have a 50% chance of winning?
- If you only bought one ticket for \$1, what is the expected value of the ticket?

Standard(s)

CCSS.Math.Content.HSS-MD.B.5
CCSS.Math.Content.HSS-MD.B.5a

Benchmark Assessment

The benchmark assessment for this standard would be a combination of informal and formal. In the classroom, the teacher would use the “show me” or “explain your work” approach to have students demonstrate their understanding and ability.

The teacher should also provide some practice problems that require students to apply the same concepts to different situations. This would allow students to demonstrate their understanding of standard S-MD.B.5, especially S-MD.B.5a.

HSS-MD.B.5, HSS-MD.B.5b

Once students have an understanding of how to find the expected value and probability of winning a game of chance, they can use this knowledge to formulate strategies to increase the chances of winning. What's more, strategies can be evaluated and compared on the basis of expected values.^{S-MD.B.5b} This can be particularly effective because students can use mathematical-based evidence to justify their approaches.

A simple activity such as rolling of dice can be used to engage students in comparing and evaluating strategies. For example, a game can be devised where values are assigned to the possible combinations (2-24 for two dice). Students would have to decide and justify the amount of money they would wager for a specific roll. To generalize the concept, add more die and assign more values. Then, add more variables such as weighted, unfair dice to manipulate the outcome. With making students aware of loaded dice, have them record totals over a number of consecutive rolls and conjecture what sum would give the best probability of success. Students can see how additional variables can influence the expected value of a scenario.

Another example would be to use compare a low versus a high-deductible automobile insurance policy*. Students can use reasonable hypotheses to decide the chances of being in a minor or major accident. This is more of a real life application; students will eventually have to decide on their own insurance, and being able to reasonably estimate their chances of injury or accident could influence their decision.

These types of problems present students with a scenario where there is not always a clear strategy. In order to arrive at strategy or answer that produces the desired expected value, students have to evaluate and compare approaches using mathematical reasoning.^{S-MD.B.5b}

*Example Problem: Deductibles

1a) You just got a car for your birthday! But now you have to decide which insurance plan to buy. Given that you drive roughly 20 miles on the freeway to school during the week, and 15 miles through downtown on the weekends to your part-time job, which of the three insurance policies below would you choose. Why? (Use NHTSA data to inform your decision. Info can be found at:

<http://www.nrd.nhtsa.dot.gov/departments/nrd-30/nca/STSI/USA%20WEB%20REPORT.HTM>.)

Plan 1: "Catastrophic Insurance"

- \$35/mo.
- 50% coinsurance for minor accidents
- 0% coinsurance for major accidents
- \$5,000 deductible

Plan 2: "Balanced Protection"

- \$70/mo.
- 50% coinsurance for minor accidents
- 25% coinsurance for major accidents.
- \$2,000 Deductible

Plan 3: "Everyday Coverage"

- \$125/mo.
- 15% coinsurance for minor accident
- 50% coinsurance for major accidents
- \$750 deductible

1b) What plan would you choose if you were planning on moving closer to school so that you could walk to school and drive 10 miles to work? Why?

Standard(s)

CCSS.Math.Content.HSS-MD.B.5
CCSS.Math.Content.HSS-MD.B.5b

Benchmark Assessment

The benchmark assessment for this standard should be relatively informal. The teacher can use student justifications and reasoning during the classroom activities. If students can use reasonable estimations of the influence of added variables, especially after data is recorded, then it can be assured that students understand the concepts of standard
CCSS.Math.Content.HSS-MD.B.5b.

HSS-MD.B.6

Perhaps the most immediately useful aspect of probability for students is to be able to establish fairness in decision making.^{S-MD.B.6} This most often relates to the idea of and absence of bias or randomness. Being able to ensure the absence of bias enables a fair decision to be made.

In previous steps in this learning progression, students have already used dice, coins, or other random number generators to model fairness, but it should be noted that there can never be true randomness or fairness. Deviation will practically always be present and should be accounted for. In some cases, the factors that influence fairness are more evident and measureable. Students should be able to identify and compensate for these variables when making decisions based on probabilities.

To illustrate this point, various methods of “fair,” or “random” scenarios can be presented. For example, the teacher can present one situation, such as assigning people a lifetime salary, and use various methods to make the choice. Drawing cards, lots or straws, rolling dice, or flipping a coin could be ways to make the decisions. Students should discuss the merits of each of the decision-making systems and what makes one more or less fair. These kinds of scenarios can also be presented as practice problem* where students not only must find the most correct answer, but explain why their answer is best.

Students then can be charged with creating their own scenarios, one that is biased (either slightly or noticeably), and one that is as fair as possible. Students should keep notes on the outcomes in order to decide if their system achieves the level of fairness that they designed it for.

These kind of activities, implemented and conducted mathematical-based approaches and discussion allows students to connect the mathematical concepts of probability fairness and randomness with intuitive models.

*Example Problems: Practice Problems

1. Which of the following decision-making systems if the most fair? Why is your answer the most fair? What aspects influence the fairness of the other systems?
 - a) Flipping a fair coin
 - b) Flipping a fair coin and using a random number generator
 - c) Flipping a fair coin and rolling dice
 - d) Flipping a fair coin and drawing cards
2. If you wanted to split a group of 20 people into teams for a game,
 - a) What variables would you need to consider to ensure the teams are as fair as possible?
 - b) Suppose the game is basketball, how does this influence your team-making strategy?
 - c) If the game is monopoly, and the goal is to have one team go bankrupt as quickly as possible, how would it influence your team-making strategy?
 - d) Suppose you don't know any of the 20 people, you don't know the game, and you don't know the prize or goal, how would divide the teams as fair as possible? Why?
3. There is one slice of pizza left between you and your friend. What is the fairest way to decide who gets the last slice? Why? (assume all other variable are even)

Standard(s)

CCSS.Math.Content.HSS-MD.B.6

Benchmark Assessment

When students design their own decision-making systems of “random” number generators, they are taking notes on the merits of their systems. When students are able to design and explain why a decision-making model accomplishes the desired level of deviation, then it will demonstrate their understanding of standard S-MD.B.6.

HSS-MD.B.7

The next step in this learning progression involves students translating the intuitive models and mathematical formulas into an understanding of how to apply them. This requires students to analyze decisions and strategies using mathematical probability and statistical concepts.^{S-MD.B.7}

The purpose is not to have students simply memorize formulas. Instead, it is to have students be able to recognize when a certain approach or strategy should be used given the scenario, and to be able to explain why that method is most appropriate. Being able to use logic and mathematical reasoning to arrive at a solution demonstrates a deeper understanding of the concepts that underlie a problem, rather than just memorization of formulas.

This kind of analysis can be illustrated by product or medical testing where the probabilities of desired outcomes must fall within a given confidence interval. Even though students might not know the complex mathematics that go into this kind of statistics, they can still analyze possible decisions using the probability concepts that have already been covered.^{S-MD.B.7} To accomplish this, students can analyze empirical data such as the probability of contracting a disease. Suppose that people who are infected with virus XX will test positive 97.6% of the time, and that people who are not infected with virus XX test negative only 85.5% of the time. Additionally, in a given area, 1 in 5,000 test positive. Students can be asked to find the probability that a person who tested negative for virus XX is actually free of infection, and to use this information to inform a decision regarding preventative treatment.

Students can use various approaches to solve this problem, but the key is that they are able to explain their reasoning. After learning the skills and procedures to analyze a scenario in this manner, students can analyze similar decisions-making processes in other scenarios, as presented in practice problems*. Still, the emphasis should be on explaining the mathematical reasoning behind a conclusion, not just giving an answer.

*Example Problems: Practice Problems

1. The odds of being struck by lightning in the U.S. in a given year is 1 in 700,000, and the odds of being struck in your lifetime average 1 in 3,000. Based on probability, if you and 2,999 other people are standing in a field in Kansas during a thunderstorm, would only one of you be struck by lightning? Why or why not? (Explain your reasoning)
2. The product safety analyst of a light bulb manufacturer concluded that about one out of every 100,000 bulbs will be faulty and explode after only 100 hours of use rather than the advertised 10,000 hours. It will cost the company about a million dollars to update equipment to fix the problem to an acceptable level. If the price per light bulb is increased by \$0.10, and the company averages selling 2.5 million bulbs a year, would making the required updates be cost-effective? Why? (Explain your reasoning)

Standard(s)

CCSS.Math.Content.HSS-MD.B.7

Benchmark Assessment

The benchmark assessment would be successful completion of practice problems.* The problems must be designed and assessed in such a way that students explain their reasoning and analysis of decisions and strategies using probability concepts.^{S-MD.B.7}