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Grade 10-12

Text: Integrated Mathematics 3

**High School: Number and Operations: Perform Operations on Matrices and use Matrices in Applications.**

Starting from Kindergarten students are introduced to different kinds of numbers and operations. The begin with counting numbers 1,2,3 and eventually add 0 to their numbers thus giving them access to the whole numbers. Then as they move closer to middle school the students begin working with fractions. As they move through these numbers they begin seeing negative fractions as well establishing the rational numbers for these students. Near the end of middle school these students combine the rational numbers with the irrational numbers creating the real numbers. Then in High school these students see another number system consisting of complex numbers.

An important note for this is that through all of these different number systems the same operations of addition, subtraction, multiplication, and division can be done. This means that these number systems are interconnected and likely have different properties in common as well. Properties like the associative or commutative properties that allow the students to manipulate the numbers to find solutions to problems. Students use these properties alongside of addition and subtraction, and as students move through the grade levels they begin to multiply and divide, eventually working with exponents and roots.

Another numbers and operations that students are introduced to in high school is matrices. A matrix is an array of numbers, symbols, or expressions arranged in rows and columns. These are used to solve various kinds of real world problems.

**Perform operations on matrices and use matrices in applications.**

There are games that are played that can be represented by a matrix. The prisoner’s dilemma goes thusly: two criminals are arrested and detained in separate cells, the police know they cannot imprison both criminals on the charge they were arrested for. However, they can sentence each criminal to 1 year each. Each criminal is offered a bargain, if they testify against the other criminal they will not be

|  |  |  |
| --- | --- | --- |
|  | Criminal 2 remain silent | Criminal 2 testify |
| Criminal 1 remain silent | 1 year \1 year | 3 year \ 0 years |
| Criminal 1 testify | 0 years \ 3 years | 3 year \ 3 year |

[CCSS.Math.Content.HSN-VM.C.6](http://www.corestandards.org/Math/Content/HSN/VM/C/6) (+)

 Use matrices to represent and manipulate data, e.g., to represent payoffs or incidence relationships in a network.

Criminal matrix $\left[\begin{matrix}1&3\\0&3\end{matrix}\right]$

sentenced for the previous crime and the other criminal will serve 3 years. If both criminals testify against the other then both receive 3 years. Now what would happen to the game if the stakes were upped? Let’s say we doubled the years each criminal would receive. Then it would look something like this: Criminal 1 Matrix looks like $\left(\begin{matrix}1&3\\0&3\end{matrix}\right)$ while criminal 2 matrix looks like $\left(\begin{matrix}1&3\\0&3\end{matrix}\right)$ as well. So to double it would look like 2\*$\left(\begin{matrix}1&3\\0&3\end{matrix}\right)=\left(\begin{matrix}2&6\\0&6\end{matrix}\right)$ where each value in the Matrix is multiplied.

Matrix addition and subtraction are the same as adding and subtracting regular numbers with only one difference. The difference is that matrices can only be added to or subtracted from other matrices of the same size. This is done by adding the numbers in the corresponding position together.

Matrix multiplication on the other hand is a different story. In addition and subtraction the corresponding values are added or subtracted, multiplication is not so simple. In order to multiply a matrix to another matrix, it is done by multiplying the row of the first matrix into the column of the second matrix. These two values are then added to each other and placed in the appropriate position. For example, if the first row of matrix A is multiplied to the first column of matrix B then the first value gets multiplied by the first and second gets multiplied to second. These are then added to each other and placed in the first row, first column of the newly formed matrix C.

The associative property states that if you have numbers arranged as follows: $1\*\left(2\*3\right)$ You can rearrange the parenthesis and get$ \left(1\*2\right)\*3$. Both equations yield the same answer of 6. It does not matter if 2 is multiplied to 3 and then multiplied to 1 or 1 is multiplied to 2 and then to 3. We could have used any numbers here for 1 2 and 3, thus the multiplication is associative over the real numbers.

[CCSS.Math.Content.HSN-VM.C.7](http://www.corestandards.org/Math/Content/HSN/VM/C/7) (+)

Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled.

$\left[\begin{matrix}1&3\\0&3\end{matrix}\right]+\left[\begin{matrix}1&3\\0&3\end{matrix}\right]=\left[\begin{matrix}1+1&3+3\\0+0&3+3\end{matrix}\right]=\left[\begin{matrix}2&6\\0&6\end{matrix}\right]$ Which is identical to the doubled matrix, likewise, $\left[\begin{matrix}1&3\\0&3\end{matrix}\right]-\left[\begin{matrix}1&3\\0&3\end{matrix}\right]=\left[\begin{matrix}1-1&3-3\\0-0&3-3\end{matrix}\right]=\left[\begin{matrix}0&0\\0&0\end{matrix}\right]$ since the matrix was subtracted from its self, each value became 0.

[CCSS.Math.Content.HSN-VM.C.8](http://www.corestandards.org/Math/Content/HSN/VM/C/8) (+) Add, subtract, and multiply matrices of appropriate dimensions.

 $\left(\begin{matrix}1&3\\0&3\end{matrix}\right)\*\left(\begin{matrix}1&3\\0&3\end{matrix}\right)=\left(\begin{matrix}(1\*1+3\*0)&(1\*3+3\*3)\\(0\*1+3\*0)&(0\*3+3\*3)\end{matrix}\right)=\left(\begin{matrix}1&12\\0&9\end{matrix}\right)$ this means that the answer depends on which matrix is first and which matrix is second.

[CCSS.Math.Content.HSN-VM.C.8](http://www.corestandards.org/Math/Content/HSN/VM/C/8) (+) Add, subtract, and multiply matrices of appropriate dimensions.

[CCSS.Math.Content.HSN-VM.C.9](http://www.corestandards.org/Math/Content/HSN/VM/C/9) (+) Understand that, unlike multiplication of numbers, matrix multiplication for square matrices is not a commutative operation, but still satisfies the associative and distributive properties.

What would the Associative property look like on matrix multiplication?

Is$ \left[\begin{matrix}1&3\\0&3\end{matrix}\right]\*\left(\left[\begin{matrix}2&1\\1&2\end{matrix}\right]\*\left[\begin{matrix}1&2\\2&1\end{matrix}\right]\right)$ the same as

 $\left(\left[\begin{matrix}1&3\\0&3\end{matrix}\right]\*\left[\begin{matrix}2&1\\1&2\end{matrix}\right]\right)\*\left[\begin{matrix}1&2\\2&1\end{matrix}\right]$?

 The first one would start with$ \left[\begin{matrix}2&1\\1&2\end{matrix}\right]\*\left[\begin{matrix}1&2\\2&1\end{matrix}\right]=\left[\begin{matrix}4&5\\5&4\end{matrix}\right]$.

Then $\left[\begin{matrix}1&3\\0&3\end{matrix}\right]\*\left[\begin{matrix}4&5\\5&4\end{matrix}\right]=\left[\begin{matrix}19&17\\15&12\end{matrix}\right]$.

Now the second way starts with$ \left[\begin{matrix}1&3\\0&3\end{matrix}\right]\*\left[\begin{matrix}2&1\\1&2\end{matrix}\right]=\left[\begin{matrix}5&7\\3&6\end{matrix}\right]$.

Then goes to $\left[\begin{matrix}5&7\\3&6\end{matrix}\right]\*\left[\begin{matrix}1&2\\2&1\end{matrix}\right]=\left[\begin{matrix}19&17\\15&12\end{matrix}\right]$.

Thus matrix multiplication is associative.

Likewise the distributive property says that given the equation $2\*(3+1)$ you can

multiply the 2 into the 3 and the 1 before adding to get $(2\*3+2\*1)$ since the order of operations state multiplication before addition both equations come to the same answer of 8.

The Commutative property says given an equation 2\*3 is the same as 3\*2 as both yield the answer of 6. These 3 properties apply to the real numbers over multiplication. Some of these also apply to Matrix operations, one does not.

 As demonstrated matrix multiplication satisfies the associative and distributive properties, however it does not satisfy the commutative property. Ex, $\left[\begin{matrix}1&3\\0&3\end{matrix}\right]\*\left[\begin{matrix}2&1\\1&2\end{matrix}\right]=\left[\begin{matrix}5&7\\3&6\end{matrix}\right]$ However $\left[\begin{matrix}2&1\\1&2\end{matrix}\right]\*\left[\begin{matrix}1&3\\0&3\end{matrix}\right]=\left[\begin{matrix}2&9\\1&9\end{matrix}\right]$ Since the first answer is not the same as the second this shows that matrix multiplication is not commutative.

In normal addition there is a number that can be added to another without changing that number namely the number 0 aptly named the additive identity. The same can be said of normal multiplication namely the number 1 aptly named the multiplicative identity. This is also true with addition and multiplication with matrices. However since it is a matrix, it is not as simple as a single number like 0 or 1.

That being said this means that for most matrices there exist a matrix, let’s say the inverse such that matrix A times matrix A inverse, denoted as A-1, or A\*A-1= the identity namely $\left[\begin{matrix}1&0\\0&1\end{matrix}\right]$. Since this is true there must be a way to find the inverse of a matrix. This is done by taking the determinant, found by cross multiplying the top right with the bottom left and subtracting it with the multiplication of the bottom left and the top right, this being for a 2x2 matrix. There is an equation that will always give an inverse to any 2x2 matrix, this equation is $\frac{1}{determinant}\*MatrixA=A^{-1}$. This means that the determinant must be nonzero in order for there to be a working inverse, since the determinant is in the denominator. This ultimately means that if a matrix has a

[CCSS.Math.Content.HSN-VM.C.9](http://www.corestandards.org/Math/Content/HSN/VM/C/9) (+) Understand that, unlike multiplication of numbers, matrix multiplication for square matrices is not a commutative operation, but still satisfies the associative and distributive properties.

What about the Distributive property?

Will $\left[\begin{matrix}1&3\\0&3\end{matrix}\right]\*\left(\left[\begin{matrix}2&1\\1&2\end{matrix}\right]+\left[\begin{matrix}1&2\\2&1\end{matrix}\right]\right)$ be the same

 as $\left[\begin{matrix}1&3\\0&3\end{matrix}\right]\*\left[\begin{matrix}2&1\\1&2\end{matrix}\right]+\left[\begin{matrix}1&3\\0&3\end{matrix}\right]\*\left[\begin{matrix}1&2\\2&1\end{matrix}\right]$?

The first way starts with $\left[\begin{matrix}2&1\\1&2\end{matrix}\right]+\left[\begin{matrix}1&2\\2&1\end{matrix}\right]=\left[\begin{matrix}3&3\\3&3\end{matrix}\right]$.

Then is multiplied by$ \left[\begin{matrix}1&3\\0&3\end{matrix}\right]\*\left[\begin{matrix}3&3\\3&3\end{matrix}\right]=\left[\begin{matrix}12&12\\9&9\end{matrix}\right]$.

The second way starts with $\left[\begin{matrix}1&3\\0&3\end{matrix}\right]\*\left[\begin{matrix}2&1\\1&2\end{matrix}\right]=\left[\begin{matrix}5&7\\3&6\end{matrix}\right]$.

Then $\left[\begin{matrix}1&3\\0&3\end{matrix}\right]\*\left[\begin{matrix}1&2\\2&1\end{matrix}\right]=\left[\begin{matrix}7&5\\6&3\end{matrix}\right]$.

Finally $\left[\begin{matrix}5&7\\3&6\end{matrix}\right]+\left[\begin{matrix}7&5\\6&3\end{matrix}\right]=\left[\begin{matrix}12&12\\9&9\end{matrix}\right]$.

Thus matrix multiplication is distributive.

[CCSS.Math.Content.HSN-VM.C.10](http://www.corestandards.org/Math/Content/HSN/VM/C/10) (+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.

The additive identity for a 2x2 matrix is $\left[\begin{matrix}0&0\\0&0\end{matrix}\right]$ so that

 $\left[\begin{matrix}1&2\\2&1\end{matrix}\right]+\left[\begin{matrix}0&0\\0&0\end{matrix}\right]=\left[\begin{matrix}1+0&2+0\\2+0&1+0\end{matrix}\right]=\left[\begin{matrix}1&2\\2&1\end{matrix}\right]$ therefore $\left[\begin{matrix}0&0\\0&0\end{matrix}\right]$ must be the additive identity for a 2x2 matrix.

The multiplicative identity for a 2x2 matrix is $\left[\begin{matrix}1&0\\0&1\end{matrix}\right]$ so that $\left[\begin{matrix}1&0\\0&1\end{matrix}\right]\*\left[\begin{matrix}1&2\\2&1\end{matrix}\right]= \left[\begin{matrix}1\*1+0\*2&1\*2+0\*1\\0\*1+1\*2&0\*1+1\*1\end{matrix}\right]=\left[\begin{matrix}1&2\\2&1\end{matrix}\right]$Thus $\left[\begin{matrix}1&0\\0&1\end{matrix}\right]$ is the multiplicative inverse of a 2x2 matrix. This is true for any size so long as the diagonal starting in the top right corner is 1.

There is an equation that will always give an inverse to any 2x2 matrix this equation is $\frac{1}{determinant}\*MatrixA=A^{-1}$. This means that the determinant must be nonzero in order for there to be a working inverse, since the determinant is in the denominator.

multiplicative inverse, the determinate must be nonzero.

One way of writing a vector is as a 1x2 or 2x1 matrix such as$\left(\genfrac{}{}{0pt}{}{1}{1}\right) or \left(\begin{matrix}1&1\end{matrix}\right)$ in this way we can actually multiply these vectors together thus getting a transformation of vector.

Various matrices can be regarded as vectors and transformations of vectors. This allows for easy representation of the transformation of multiple vectors.

 In Geometry students learn how to do transformations of shapes. They rotate, slide, and flip various shapes along different focal points. Matrices can actually be used to represent these kinds of transformations in the standard xy-plane. This can then be applied to different planes that can be used, such as the complex plane or even the xyz-plane. This is done by establishing the axis of the plane to be worked in, such as the x-axis and y-axis in the xy-plane. Then by establishing the matrix representation of these axes, one can begin transforming these vector matrices to show the various transformations of shapes in the plane.

[CCSS.Math.Content.HSN-VM.C.11](http://www.corestandards.org/Math/Content/HSN/VM/C/11) (+) Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.



 d c b a e

Matrix M$=\left[\begin{matrix}0&8&8\\0&0&6\end{matrix} \begin{matrix}4&0\\10&6\end{matrix}\right]$

What would $\left[\begin{matrix}1&2\\2&1\end{matrix}\right]$M be?

[CCSS.Math.Content.HSN-VM.C.12](http://www.corestandards.org/Math/Content/HSN/VM/C/12) (+) Work with 2 × 2 matrices as a transformations of the plane, and interpret the absolute value of the determinant in terms of area.

The standard matrix for the linear transformation*T*:*R*2*R*2 that rotates vectors by an angle θ is

A=$\left[\begin{matrix}cosθ&-sinθ\\sinθ&cosθ\end{matrix}\right]$ This is easily derived by noting that

T$\left(\left[\begin{matrix}1\\0\end{matrix}\right]\right)=\left[\begin{matrix}cosθ\\sinθ\end{matrix}\right]$

T$\left(\left[\begin{matrix}0\\1\end{matrix}\right]\right)=\left[\begin{matrix}-sinθ\\cosθ\end{matrix}\right]$

