Illustrative Mathematics
HSF-TF.C - Proving Trigonometric Identities
Alignment 1: TF.C.8

Reciprocal Identities

$$\sin\theta = \frac{1}{\csc\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{1}{\sec\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Quotient Identities

$$\frac{\sin\theta}{\cos\theta} = \tan\theta$$

$$\frac{\cos\theta}{\sin\theta} = \cot\theta$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

Prove that the Left Hand Side (LHS) of the equation is equal to the Right Hand Side (RHS) using the quotient, reciprocal and pythagorean identities of trigonometry.

a.
$$\frac{\cos^2 x - \sin^2 x}{1 - \tan^2 x} = \cos^2 x$$

b.
$$(\csc^2 x - 1)(\sec^2 x \cdot \sin^2 x) = 1$$

c.
$$\sin x - \sin x \cos^2 x = \sin^3 x$$

d.
$$\cot x + \tan x = \sec x \csc x$$

Commentary

This task asks students to prove the Left Hand Side (LHS) of the equation to the Right Hand Side (RHS) using the quotient, reciprocal and pythagorean identities of trigonometry. Students can complete the task using different steps as long as each step is valid using trigonmetric identities and algebraic processes. This task will develop students' procedural fluency in memorizing and manipulating trigonometric identities.

A teacher who uses this problem as a classroom task should ask students to present different solutions to the same problem.

Solution:

a. We have

$$\frac{\cos^2 x - \sin^2 x}{1 - \tan^2 x} = \cos^2 x$$

$$\frac{\cos^2 x - \sin^2 x}{1 - \frac{\sin^2 x}{\cos^2 x}} = \cos^2 x \qquad \left(\tan^2 x = \frac{\sin^2 x}{\cos^2 x}\right)$$

$$\frac{\cos^2 x - \sin^2 x}{\frac{\cos^2 x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x}} = \cos^2 x \qquad \text{Common denominator of } \cos^2 x$$

$$\frac{\cos^2 x - \sin^2 x}{\frac{\cos^2 x - \sin^2 x}{\cos^2 x}} = \cos^2 x$$

$$(\cos^2 x - \sin^2 x) \cdot \left(\frac{\cos^2 x}{\cos^2 x - \sin^2 x}\right) = \cos^2 x \qquad \text{Multiply by reciprocal of } \frac{\cos^2 x - \sin^2 x}{\cos^2 x}$$

$$\cos^2 x = \cos^2 x$$

b. We have

$$(\csc^2 x - 1)(\sec^2 x \cdot \sin^2 x) = 1$$

$$(\csc^2 x - 1) \cdot \left(\frac{1}{\cos^2 x} \cdot \sin^2 x\right) = 1 \qquad \left(\sec^2 x = \frac{1}{\cos^2 x}\right)$$

$$(\csc^2 x - 1) \cdot \left(\frac{\sin^2 x}{\cos^2 x}\right) = 1$$

$$(\cot^2 x) \cdot \left(\frac{\sin^2 x}{\cos^2 x}\right) = 1 \qquad \text{From pythagorean identity } \cot^2 x + 1 = \csc^2 x$$

$$(\cot^2 x) \cdot (\tan^2 x) = 1$$

$$\left(\frac{1}{\tan^2 x}\right) \cdot (\tan^2 x) = 1$$

$$1 = 1$$

c. We have

$$\sin x - \sin x \cos^2 x = \sin^3 x$$

$$\sin x\left(1 - \cos^2 x\right) = \sin^3 x \qquad \text{Factoring } \sin x \text{ from both terms}$$

$$\sin x \cdot (\sin^2 x) = \sin^3 x \qquad \text{From pythagorean identity } \cos^2 x + \sin^2 x = 1$$

$$\sin^3 x = \sin^3 x$$

d. We have

$$
\begin{aligned}
\cot x + \tan x &= \sec x \csc x \\
\frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} &= \sec x \csc x \\
\frac{\cos^2 x}{\sin x \cos x} + \frac{\sin^2 x}{\sin x \cos x} &= \sec x \csc x && \text{Common denominator of } (\sin x \cos x) \\
\frac{\cos^2 x + \sin^2 x}{\sin x \cos x} &= \sec x \csc x \\
\frac{1}{\sin x \cos x} &= \sec x \csc x && \text{From pythagorean identity } \cos^2 x + \sin^2 x = 1 \\
\left(\frac{1}{\sin x}\right) \cdot \left(\frac{1}{\cos x}\right) &= \sec x \csc x && \text{Splitting up fraction into two factors} \\
\sec x \csc x &= \sec x \csc x
\end{aligned}
$$