Learning Progression: Complex Numbers

High School Algebra

<u>Demographics:</u> This learning progression is crafted to take place within a high school Algebra class of 25 students comprised of sophomores and freshman. The class is arranged to have four students sitting together at tables to form groups. The lesson is supplemented by material from the high school math textbook, “Algebra 2” by Holt, Rinehart, and Winston. The students have prerequisite knowledge concerning the execution of the quadratic formula, familiarity with fractions, the distributive rule, and basic knowledge of a conjugate.

<u>Addressed CCSS for Mathematics:</u>

Perform arithmetic operations with complex numbers.
CCSS.MATH.CONTENT.HSN.CN.A.1
Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.
CCSS.MATH.CONTENT.HSN.CN.A.2
Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.
CCSS.MATH.CONTENT.HSN.CN.A.3
(+) Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers.

<u>Addressed Mathematical Practices:</u>

CCSS.MATH.PRACTICE.MP2 Reason abstractly and quantitatively.

CCSS.MATH.PRACTICE.MP3 Construct viable arguments and critique the reasoning of others.

<u>Learning Progression Overview:</u>

The lesson will open up with a brief recap to refresh the students on the prerequisite knowledge that is required. After this, the instructor is to present the concept of an imaginary number. The presentation follows the same format as described in the textbook, “Algebra 2”. To put imaginary numbers into context, the instructor will provide a quadratic equation, such that an imaginary number is produced, for the students to solve. Upon deriving the complex number, the instructor will introduce the properties of the imaginary number, i and complex numbers. The students will then transition into their first entry task of performing operations in complex numbers, beginning with addition and subtraction. Succeeding this, the students will learn proceed to their second entry task of multiplying complex numbers, employing their knowledge of the distributive property. The third entry task will center on the utilization of the conjugate to simplify instances of a number being divided by a complex number. Throughout, the instructor is to prompt the class with short “checks” in the form of verbal questions to assess the progress of the class. Finally, the students will be administered an assessment to gauge their overall mastery of the material.

Supports:

Math stresses problem solving, which naturally leads to group collaboration. To provide support to students who may struggle during the learning progression, students will have access to each other to work in small groups. Through this, the students may find assistance from their peers. Peer-instruction of this nature is beneficial for reasons two-fold: For the students that have a grasp of the material, they will have the opportunity to teach their peers, which reinforces their knowledge. For the students who are struggling with the material, their peers who are assisting them may be able to deliver the information in a vernacular and format that is more suited and understandable to them than the information relayed by the instructor. Should peer instruction be insufficient, the instructor may also provide (on paper handouts or written on a medium that is visible to the class) general rules and properties of complex numbers for the class, such as $i^2 = -1$. On top of this, after the deliverance of each new concept, the teacher will produce hinge questions to gather evidence of class understanding to give direction to the lesson.

Learning Progression:

The lesson will open up with a recap to refresh the students on the prerequisite knowledge that is required. To do this, the students are to be provided with a quadratic equation to solve via the quadratic formula. After a brief time, the instructor will call upon students at random to guide the instructor into solving the problem. Students are called at random for the purpose of stumping some students to make the review into a learning opportunity. For the students who have difficulty recalling material on quadratics, the instructor will make points to employ the Socratic Method. This is done by asking the class questions **[1]**. Following the review, the instructor is to present the concept of an imaginary number**.** This is done by giving the students a quadratic equation such that the radical (defined by the book as the **discriminant**) yields a negative **[2]**. Upon deriving the complex number, the instructor will introduce the properties of the imaginary number, *i* and complex numbers. Specifically, the instructor will define *i* as being equal to $\sqrt{-1}$ and complex numbers to follow the form of $a + bi$ **[3]**. To further student understanding of *i*, the instructor will ask, "If *i* is equal to $\sqrt{-1}$,what is i^2 equal to?" **[4]** Successively, the class will be prompted to discover what i^3, i^4, and i^5 are equal to **[5]**. After solving for i^5, the instructor will have the class explain why *i* and i^5 are equal **[MP2]**. Mastery of this knowledge satisfies the first Common Core standard. To provide a support for the

[1] Questions such as, "What formula do we need to solve this?", "What is the first step?", and "What is the radical going to be?"

[2] The book gives the example of $3x^2 – 7x + 5 = 0$

[3] Where *a*, *b* are real numbers.

[4] This is the first hinge question where students will have to reason **[MP2]** that when two square roots are multiplied, the result is the rooted number.

[5] $i^2 = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$

students, the instructor will write out the properties of *i* at the front of the room for the class to see and reference to **[5]**. Moving into their first entry task, the instructor will bring about the basics of mathematical operations involving complex numbers, namely, addition and subtraction **[6]**. The instructor will work out the addition problem [(-3 + 5*i*) + (7 – 6*i*)] step-by-step while narrating the process. The instructor will have the class note that *i* functions like a variable *x*, and as such, the class must only combine like-terms. Upon reaching the final solution, the students will be presented with the subtraction problem [(-3 – 8*i*) - (-2 – 9*i*)] of complex numbers to work out on their own. After a time, the instructor will ask for a volunteer to present their solution for the class, working out the problem and narrating their work. Should the solution be correct, the instructor will have the student explain why the 2 and 9*i* are positive **[7]**. At this point, the class will be open to ask any questions about the preceding example, to which the presenting student will have the opportunity to answer and justify their solution **[MP3]**. Transitioning into segment two, the instructor will pose the problem (2 + *i*)(-5 – 3*i*). **[8]** This time for the solution, the instructor will call on volunteers one at a time to give successive steps in solving the problem. Because the students should already have experience with utilizing the distributive property, it is only based on class response that the instructor may or may not do more example problems of multiplication with complex numbers. After the solution the instructor will write the F.O.I.L. method for distribution at the front of the class for the students to see and reference. Segment three will focus on conjugates and division of complex numbers. To gauge student understanding and introduce the importance of the conjugate in relation to complex numbers, the instructor will continue on multiplication and have the class solve a problem centered on multiplying a complex number by its conjugate, without telling the students that the two complex numbers are in fact, conjugates **[9]**. While the students go over the problem at their respective groups, the instructor will monitor the classroom, answering questions as needed in a manner that does not simply give the answer (Socratic Method suggested). Returning to the front, the instructor will reveal that the numbers are conjugates and that conjugates are essential for simplifying complex numbers located in the divisor. Mastery on the multiplication of complex numbers satisfies the second Common Core standard. To refresh the students on conjugates, the instructor will first ask if any of the students can tell the class what a conjugate is or what it looks like.

[6] The book provides the examples: (-3 + 5*i*) + (7 – 6*i*), (-3 – 8*i*) - (-2 – 9*i*)

[7] Specifically, distributing a negative sign into a negative sign makes a positive.

[8] Hinge question: What property should we use to multiply complex numbers?

[9] I.E. (3 + 2*i*)(3 – 2*i*)

Volunteer or not, the instructor will tell the class that the conjugate is simply changing the operation from addition to subtraction or vice-versa **[10]**. Students will learn how to utilize the conjugate to simplify instances of a number being divided by a complex number. An example will be provided featuring an integer being divided by a complex number **[11]**. The instructor will demonstrate how to simplify the problem by multiplying the numerator and denominator by the conjugate. The instructor will pause the solution at this point to ask the students why we are able to do that **[12]**. After setting up the problem to this point, the students will be set to work on their own to simplify the problem. The instructor will assess student understanding by having the students share their answers with their groups and explaining how they got their results. The instructor will then reveal the answer. Should any students have questions about the answer, the instructor will ask that the student's table mates explain the solution to them. As a formal, summative assessment, each group will be made to solve a single problem for two complex numbers under operations (addition, subtraction, multiplication, division) and present the solution for the class. The instructor will require the audience to concur or disagree with the presenters' solution and, in either case, explain why. This conclusion of the learning progression satisfies the final Common Core standard.

[10] I.E. The conjugate of a + b*i* is a - b*i*

[11] I.E. $\frac{4}{2+3i}$

[12] This is another opportunity for a hinge question. "Am I allowed to multiply the numerator and denominator by the same value?" "Does it change the value?" $\frac{2-3i}{2-3i} = 1 = \frac{4}{4}$ **:** Essentially, the class should answer to the notion that multiplying the numerator and denominator by the same value does not actually change the overall value of the problem.